ORTHOTROPIC LAMINATES ON ELASTIC FOUNDATION

by

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DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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by
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to the

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CERTIFICATE

This is to certify that the work entitled "ORTHOTROPIC LAMINATES ON ELASTIC FOUNDATION", by S. Nadimpalli has been carried out under my supervision and has not been submitted elsewhere for a degree.

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LIST OF SYMBOLS

σ_x	Normal stress component in x direction
τ_{xy}	Shear stress component in xy plane
ε_x	Normal strain component in x direction
γ_{xy}	Shear strain component in xy plane
E_L	Young's modulus in longitudinal direction
E_T	Young's modulus in transverse direction
ν_{LT}	Poisson's ratio in L-T directions
G_{LT}	Shear stress in L-T plane
$[Q]$	Stiffness matrix in principal directions
$[\overline{Q}]$	Transformed stiffness matrix in arbitrary ply orientation
$\overline{Q}_{ij}$	Component of $[\overline{Q}]$ matrix
u, v, w	Displacement components in xyz directions
u_o, v_o, w_o	Mid plane displacement components in xyz directions
ε_x^o	Mid plane normal strain in x direction
γ_{xy}^o	Mid plane shear strain in xy plane
k_x	Curvature in x direction
k_{xy}	Twist curvature in xy plane
N_x	Normal force per unit length in x direction
N_{xy}	Shear force per unit length in xy plane
M_x	Moment per unit length in x direction
M_{xy}	Twist per unit length in xy plane
A_{ij}	Components of extensional stiffness matrix [A]
B_{ij}	Components of coupling stiffness matrix [B]

D_{ij}	Components of bending stiffness matrix [D]
Ψ_x	Total shear rotation about x axis
$\{\sigma_{LT}\}$	Stress vector in principal directions
$\{\varepsilon_{LT}\}$	Strain vector in principal directions
Q_x	Total shear force due to transverse shear
ρ	Mass density
p	Normal inertia coefficient
I	Rotary inertia coefficient
ν	Isotropic Poisson's ratio
E_f	Foundation Young's modulus
ν_f	Foundation Poisson's ratio
w_{mn}	Flexibility coefficient in vertical direction due to vertical force
W_{MN}	Flexibility coefficient in vertical direction due to horizontal force
V_n	Vertical point load
H_n	Horizontal point load
$[k_f]$	Foundation stiffness matrix
$[F_f]$	Foundation flexibility matrix
$\{d\}$	Nodal displacement vector of foundation
$\{P\}$	Nodal force vector of foundation
h	Plate thickness
N_i	Shape functions
$\{u\}$	Displacement at arbitrary point within element
$\{\delta_r\}$	Nodal displacement vector
$\{\varepsilon\}$	Generalised strain vector
J	Jacobian of transformation

$\ddot{\delta}$	Second derivative with respect to time
$[M^e]$	Elemental mass matrix
$\{a\}$	Elemental inertia force vector
$[M]$	Global mass matrix
$[K]$	Global stiffness matrix
$\{\Delta\}$	Global displacement vector
$\{\bar{\Delta}\}$	Maximum global displacement vector
ω	Angular frequency
Ω	Non dimensionalised frequency

SYNOPSIS

of the
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The effects of elastic foundation on the fundamental natural frequencies of isotropic plates and orthotropic laminates are investigated. A finite element model is developed to take into account the shear rotation and rotary inertia effects. A generalised foundation model based on the concept of isotropic elastic half space is used. This model can be reduced to one parameter or two parameter models. Simply supported square plate/laminate is considered for the analysis. Effects of varying foundation stiffness, ply angles, aspect ratio and side to thickness ratios on the fundamental frequencies are investigated. The results are plotted and tabulated.

CHAPTER 1

INTRODUCTION

Rectangular plate is one of the most commonly used structural members in present day's industrialised world. In most design problems ensuring the safety of the plate by mere static analysis is often inadequate. Therefore, it requires a dynamic analysis that takes into account the time varying forces, periodic or random, as experienced in cases of rectangular plates used on ships and submarines, subjected to a tangential liquid force or those experienced by the off-shore drilling platforms due to fluid plate interaction.

Since the advent of fiber composites, composite plates and laminates are fast replacing those made of conventional metals and alloys in specialised weight sensitive applications like aerospace and marine. In general, composite materials are most ideally suited and are cost effective for structural applications where high strength to weight and high stiffness to weight ratios are required. In the recent years, development of solid propellent rocket motors, use of soft filaments in aerospace applications intensified the need for solutions of various problems of beams, plates and shells continuously supported by elastic or visco elastic media.

The fact that the free vibration analysis for fundamental natural frequencies is the first logical step for

dynamic analysis of a structure, need not be overemphasized. In the present study fundamental frequencies of a composite laminate resting on an elastic foundation are investigated.

1.1 Literature Review:

Since Kirchoff's thin plate theory, which neglects the effects of rotary inertia and transverse shear, a number of successful attempts have been made to improve the theory by introducing the above effects. One of the most widely accepted among these is Reissner's work [1] which included the effect of transverse shear deformation in the plate bending theory. Later, R.D. Mindlin [2] included the effect of rotary inertia in addition to the transverse shear effect in flexural motion of isotropic plates. Analytical solutions for static analysis of thick plates with regular geometries and boundary conditions have since been obtained by various authors, in particular D.H. Donnel et al [3], Salerno and Goldberg [4] and Carley and Langhaar [5]. Dynamic analyses of thick rectangular plates were carried out by Mindlin, Shacknow and Deresiewicz [6] which included the effects of rotary inertia and transverse shear in flexural vibrations. Srinivas and Rao [7] presented an analysis of bending, buckling and vibration of simply supported thick orthotropic rectangular plates and laminates by classical theory and compares the exact results with those obtained by thin plate theory in cases of displacements, stresses and natural frequencies. Smith [8]

analysed moderately thick rectangular plates for bending, using rectangular finite elements with nine generalised displacements at each node. Grieman and Lynn [9] developed a rectangular finite element model with transverse displacement and two orthogonal rotations as the nodal degrees of freedom. Pryor, Baker and Frederick [10] formulated finite element bending analysis of Reissner plates using rectangular elements with four nodes, each node having five nodal degrees of freedom viz., transverse displacement, two transverse shear strains and two total rotations.

In the field of finite element analysis of plate dynamics, a significant contribution is made by Rock and Hinton [11, 12], who introduced an 8-noded rectangular isoparametric element to take into account the transverse shear deformation. They also suggested different schemes for lumping masses to obtain the mass matrices that take into account the effect of rotary inertia. Reddy [13] formulated the free vibration analysis of laminated plates by finite element method using the theory postulated by Yang, Norris and Stavsky [14].

A critical study of various foundation models was presented by Kerr [15]. A finite element method to incorporate the foundation stiffness in the structure stiffness was first suggested by Cheung and Zienkiewicz [16], in which static analysis of plates and tanks resting on elastic foundations was carried out. This work was later followed up by Cheung and Nag [17] who modelled the elastic continuum of

the foundation as isotropic elastic half plane and elastic half space to analyse the linear and non linear behaviour of beams and plates on elastic foundations. Svec [18] investigated thick plates on Winkler foundation. Kameswara Rao et al investigated dynamic response of beams on generalised elastic foundations [19]. More recently, Thangambabu et al [20] carried out frequency analysis of thick orthotropic plates on Winkler foundation by finite element method using high precision triangular elements.

1.2 Objective and Scope of Present Work:

To date not much work has been published on the dynamic analysis of plates resting on elastic foundations. In the present analysis, an effort is made to investigate the effects of elastic foundation on the fundamental natural frequencies of laminates by finite element method. The study is limited to small amplitude vibrations, preserving the linearty. Effects of varying ply angle, aspect ratio, side to thickness ratio and foundation modulus on the fundamental frequencies are tabulated.

CHAPTER 2

ANALYTICAL FORMULATION OF THE PROBLEM

2.1 Introduction:

In this chapter, the concepts of anisotropy and orthotropy and their application to derive the laminate stress strain relationships are briefly introduced prior to the analytical formulation of laminate stress strain relationships. Effect of shear deformation is included in the laminate stiffness and the resultant equations of motion are presented. Formulation of the elastic foundation is presented at the end.

2.2 Stress Strain Relationship for Anisotropic Materials:

The generalised Hooke's law relating the stresses and strains for an anisotropic material with no planes of symmetry for material properties is given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{Sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad (2.1)$$

By introducing corresponding planes of symmetry of material properties in (2.1) these relationships for

different cases of anisotropy can be obtained. The stress strain relationships for an orthotropic material in principal material directions can be obtained as,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{21} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{Sym} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{21} \end{Bmatrix} \qquad (2.2)$$

which has nine independent elastic constants, free from interactions.

These elastic properties are defined in terms of engineering constants E_L, E_T, ν_{LT}, G_{LT} in principal material directions [21]. More general form of these relationships for anisotropic bodies can be found in reference [22]. The relationships defined in (2.2) can be transformed to any arbitrary orientation of fiber in case of continuous fiber lamina.

$$\{\sigma\} = [\bar{Q}]\{\varepsilon\} \qquad (2.3)$$

where $[\bar{Q}]$ is the transformed stiffness matrix.

2.3 Application to a Laminate:

2.3.1 Laminate strain displacement relationships:

A laminate is two or more laminae in arbitrary fiber orientation bonded together to act as an integral structural

element. The laminate is presumed to consist of perfectly bonded laminae with non shear deformable bonds, so that the displacements are continuous across the contact area. We assume that the normal to the middle surface remains straight, unaltered in length (i.e. $\varepsilon_z = 0$) and normal under deformation (i.e. $\gamma_{xz} = \gamma_{yz} = 0$).

The displacements u and v at any point z through laminate thickness is (Figure 2.1)

$$
\begin{aligned}
u &= u_o - z \frac{\partial w_o}{\partial x} \\
v &= v_o - z \frac{\partial w_o}{\partial y}
\end{aligned}
\tag{2.4}
$$

For small strains, the strain displacement relations are

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial u}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{aligned}
\tag{2.5}
$$

Similarly the mid plane strains in terms of displacements can be defined as

$$
\begin{aligned}
\varepsilon_x^o &= \frac{\partial u_o}{\partial x} \\
\varepsilon_y^o &= \frac{\partial v_o}{\partial y} \\
\gamma_{xy}^o &= \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}
\end{aligned}
\tag{2.6}
$$

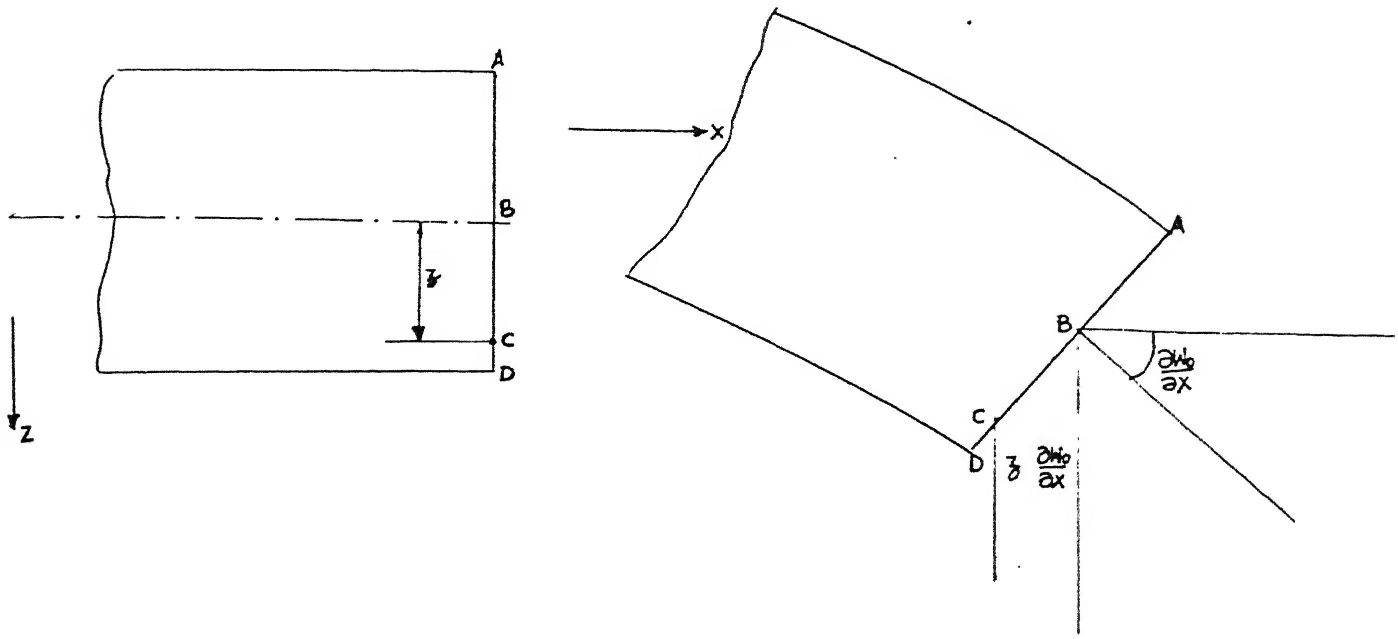

FIG 2·1 GEOMETRY OF DEFORMATION

Using (2.4) through (2.6) the total strains can be expressed in terms of midplane strains and curvatures as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^o_x \\ k^o_y \\ \gamma^o_{xy} \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{2.7}$$

where,

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = - \begin{Bmatrix} \dfrac{\partial^2 w_o}{\partial x^2} \\ \dfrac{\partial^2 w_o}{\partial y^2} \\ \dfrac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} \tag{2.8}$$

2.3.2 Resultant laminate forces and moments:

The resultant forces and moments per unit length acting on the laminate are obtained by integration of the stresses in each layer/lamina through the laminate thickness. For an N layered laminate with thickness h, the stress and moment resultants are obtained as

$$\begin{matrix} N_x \\ N_y \\ N_{xy} \end{matrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \tag{2.9}$$

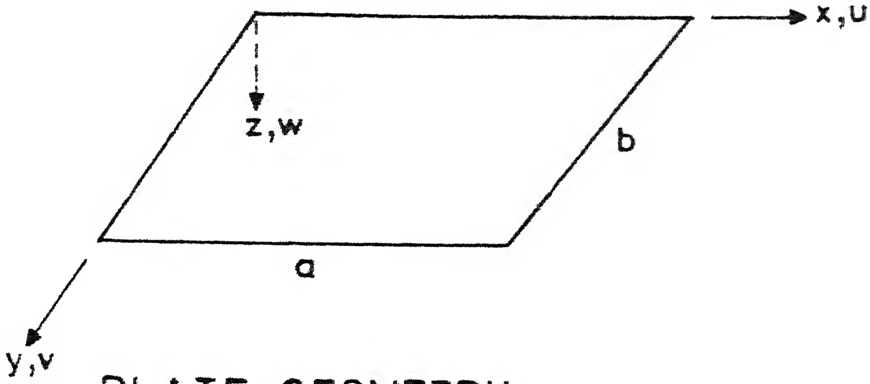

PLATE GEOMETRY

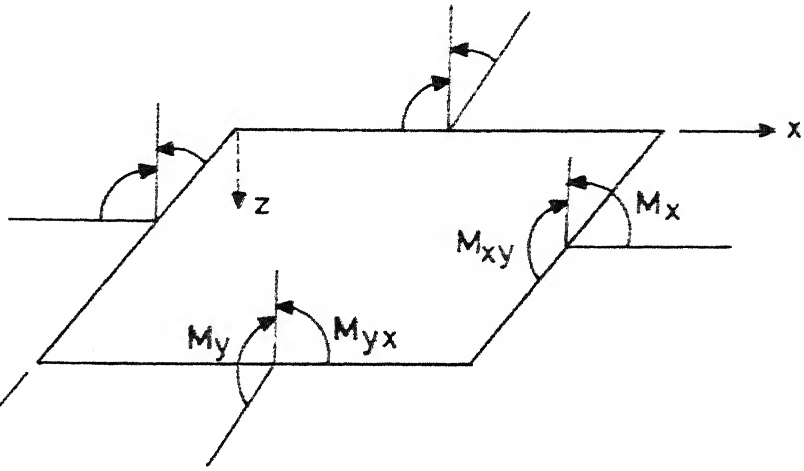

PLATE MOMENTS

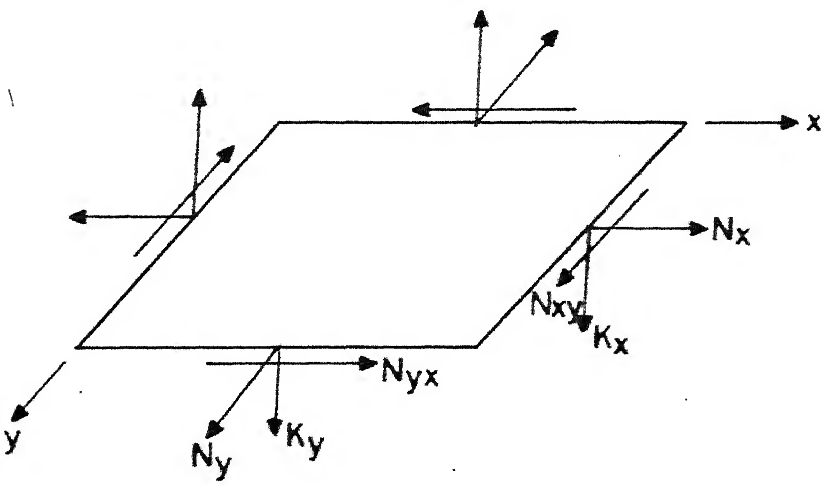

PLATE FORCES

FIG. 2.2

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\,dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\,dz \tag{2.10}$$

Using the transformed stiffness matrix of equation (2.3) in the above, the force and moment relationships are obtained as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{2.11}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

where

$$A_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k - z_{k-1})$$

$$B_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \qquad (2.12)$$

$$D_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^3 - z_{k-1}^3) , \quad \text{(Figure 2.3)}$$

2.4 Formulation of Laminate Stiffness with Shear Deformation:

2.4.1 Strain displacement relations:

A laminate with constant thickness composed of even number of thin anisotropic layers oriented at θ and $-\theta$ alternatively, is considered with the origin of the coordinate system located at the middle plane and z axis being normal to the middle plane.

By Yang, Norris and Stavskey's theory [14] the assumed displacement field in terms of midplane deformations and shear rotations

$$u = u_o(x, y, t) + z \cdot \Psi_x(x, y, t)$$

$$v = v_o(x, y, t) + z \cdot \Psi_y(x, y, t) \qquad (2.13)$$

$$w = w(x, y, t)$$

Applying the strain displacement relationships from equation (2.5) to include the shear rotations, we get,

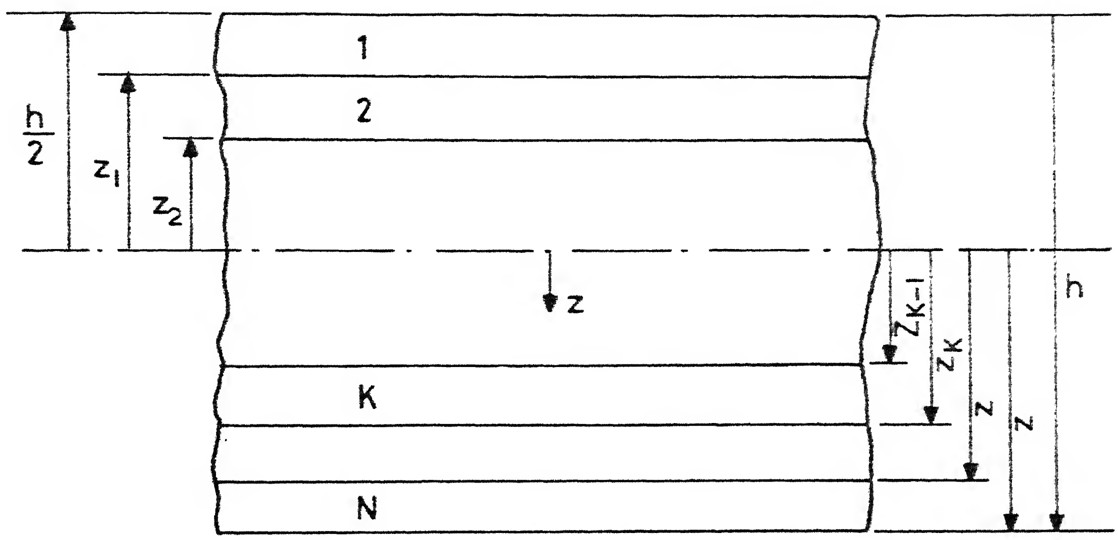

FIG. 2.3 GEOMETRY OF THE LAMINATE

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_o}{\partial x} + z\,\frac{\partial \Psi_x}{\partial x} \\
\varepsilon_y &= \frac{\partial v_o}{\partial y} + z\,\frac{\partial \Psi_y}{\partial y} \\
\varepsilon_z &= 0 \\
\gamma_{xy} &= \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\left(\frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\right) \\
\gamma_{xz} &= \Psi_x + \frac{\partial w}{\partial x} \\
\gamma_{yz} &= \Psi_y + \frac{\partial w}{\partial y}
\end{aligned}
\tag{2.14}
$$

2.4.2 Resultant laminate stiffness:

The constitutive relationships for an orthotropic layer in principal material directions L and T are given by

$$\sigma_{LT} = [Q]\varepsilon_{LT} \tag{2.15}$$

where

$$
[Q] = \begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\
Q_{12} & Q_{22} & 0 & 0 & Q_{26} \\
0 & 0 & Q_{44} & Q_{45} & 0 \\
0 & 0 & Q_{45} & Q_{55} & 0 \\
Q_{16} & Q_{26} & 0 & 0 & Q_{66}
\end{bmatrix} \tag{2.16}
$$

with Q_{ij}, $i = 1, 2, 6$

$j = 1, 2, 6$ – representing the in plane terms

and Q_{ij}, $i = 4, 5$

$j = 4, 5$ – representing the coupled shear rotation terms

Now, using the relationships for resultant forces and moments for a laminate from equations (2.4) through (2.10), we can express the constitutive equation (2.14) in terms of forces and displacements as follows

$$\begin{Bmatrix} N_1 \\ N_2 \\ Q_y \\ Q_x \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & 0 & 0 & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{44} & A_{45} & 0 & 0 & 0 & 0 \\ & & & A_{55} & 0 & 0 & 0 & 0 \\ & & & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & & & D_{11} & D_{12} & D_{16} \\ & \text{Sym} & & & & & D_{22} & D_{26} \\ & & & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial w}{\partial y} + \Psi_y \\ \frac{\partial w}{\partial x} + \Psi_x \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \\ \frac{\partial \Psi_x}{\partial x} \\ \frac{\partial \Psi_y}{\partial y} \\ \frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x} \end{Bmatrix} \quad (2.17)$$

We can assume that the shear rotation terms are independent of each other and make $A_{45} = 0$, i.e. the coupling between Q_x and Q_y is neglected.

2.4.3 Equations of motion:

The equations of motion are obtained by writing the expressions for the kinetic energy T and potential energy V and taking the first variation of the Lagrangian L = T - V, as

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = p \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial N_2}{\partial y} + \frac{\partial N_6}{\partial x} = p \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = p \frac{\partial^2 w}{\partial t^2} \tag{2.18}$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_x = I \frac{\partial^2 \Psi_x}{\partial t^2}$$

$$\frac{\partial M_2}{\partial y} + \frac{\partial M_6}{\partial x} - Q_y = I \frac{\partial^2 \Psi_y}{\partial t^2}$$

where $p = \int_{-h/2}^{h/2} \rho \, dz$

and $I = \int_{-h/2}^{h/2} \rho \, z^2 \, dz$

2.5 Formulation of Elastic Foundation:

2.5.1 Foundation flexibility and stiffness matrices:

Boussinesq [23] formulated the deflection formulae for isotropic elastic half space for point loads. Treating the forces acting at nodes as equivalent uniform pressure acting on a rectangle around the node, t.e coefficients of the foundation flexibility matrix are obtained [17] as,

for vertical pressure (Figure 2.4);

$$w_{mn} = \frac{V_n(1 - \nu_f^2)}{a \pi E_f} \left(B \sinh^{-1} \frac{1}{B} + \sinh^{-1} B - C \sinh^{-1} \frac{1}{C} - \sinh^{-1} C\right)$$

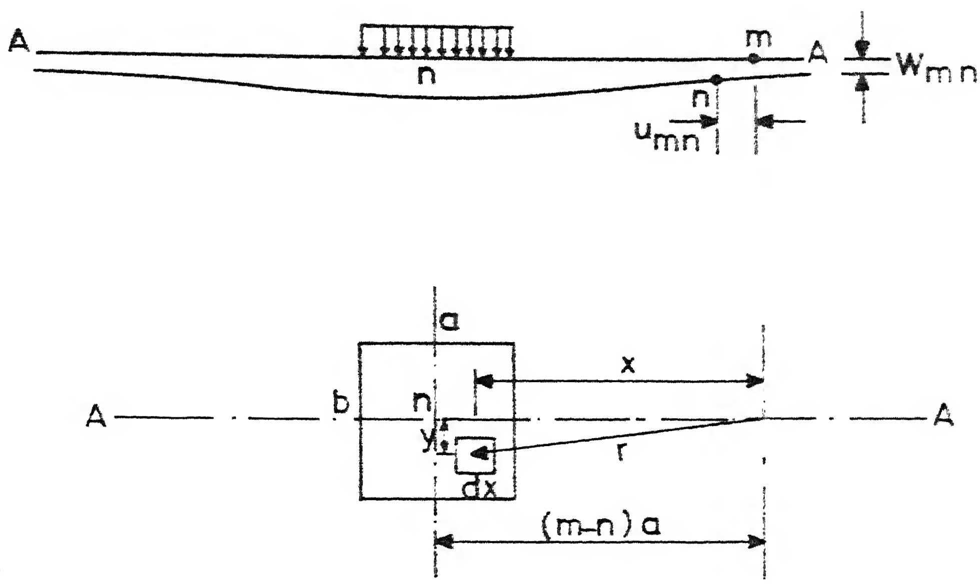

FIG.2.4 VERTICAL AND HORIZONTAL DISPLACEMENTS OF ELASTIC ISOTROPIC HALF SPACE.

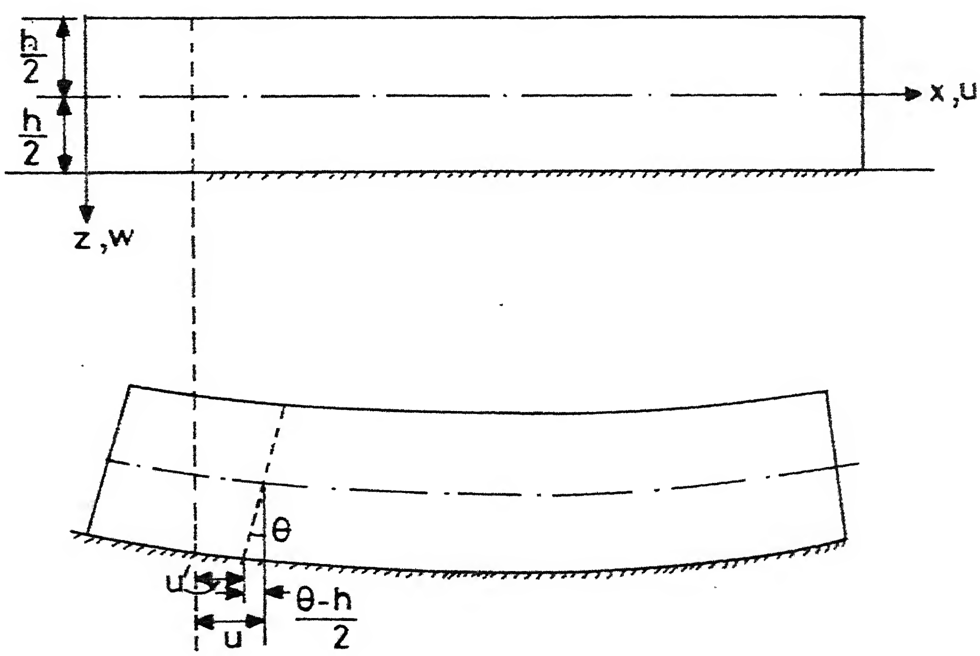

FIG.2.5 PLATE AND FOUNDATION DISPLACEMENTS.

$$u_{mn} = - \frac{V_n(1 - 2\nu_f)(1 + \nu_f)}{2\pi E_f} [B \tan^{-1} \frac{1}{B} - \ln \frac{1}{\sqrt{1 + B^2}}$$

$$- C \tan^{-1} \frac{1}{C} + \ln \frac{1}{\sqrt{1 + C^2}}] , \qquad m \neq n$$

and

$$w_{mn} = \frac{2V_n}{a \pi E_f} [B \sinh^{-1} \frac{1}{B} + \sinh^{-1} B](1 - \nu_f^2)$$

$$u_{mn} = 0 \qquad m = n$$

for horizontal pressure;

$$u_{MN} = \frac{H_n(1 - \nu_f^2)}{\pi E_f a} [B \sinh^{-1} \frac{1}{B} + \sinh^{-1} B - C \sinh^{-1} \frac{1}{C} - \sinh^{-1} C]$$

$$+ \frac{H_n \nu_f (1 + \nu_f)}{\pi E_f a} [B \sinh^{-1} B - \sinh^{-1} C], \qquad m \neq n$$

$$u_{MN} = \frac{2H_n(1 - \nu_f^2)}{\pi E_f a} [B \sinh^{-1} \frac{1}{B} + \sinh^{-1} B]$$

$$+ \frac{2H_n \nu_f (1 + \nu_f)}{\pi E_f a} \sinh^{-1} B , \qquad m = n$$

$$w_{MN} = u_{mn} \tag{2.19}$$

where,

$$B = [2(m - n) + 1] \frac{a}{b}$$

$$C = [2(m - n) - 1] \frac{a}{b}$$

a, b = sides of rectangle around the node.

The coefficients thus obtained from the equations (2.19) form the flexibility matrix of the foundation. This

matrix is inverted to obtain the stiffness matrix of the foundation.

$$\{d\} = [F_f]\{P\}$$
$$[K_f]\{d\} = \{P\} \qquad (2.20)$$
$$[K_f] = [F_f]^{-1}$$

where

$$\{d\}^t = \{d_1, d_2 \ldots . d_n\}$$
$$\{P\}^t = \{P_1, P_2 \ldots . P_n\}$$

2.5.2 Generalised foundation stiffness:

The stiffness matrix obtained in equation (2.20) is with respect to the foundation plate interface and that of the plate in general, is with respect to the neutral surface. Therefore the terms involving the horizontal pressures need to be modified to account for the plate thickness (Figure 2.5).

At any node i, the relationships between plate and foundation displacements are given by

$$w_i' = w_i$$
$$u_i' = u_i - \theta_x(\tfrac{h}{2})$$
$$v_i' = v_i - \theta_y(\tfrac{h}{2}) \qquad (2.21)$$
$$V_i' = V_i$$
$$H_i' = H_i$$
$$M_i = H_i'(\tfrac{h}{2})$$

with primed symbols representing foundation system.

Using (2.21) in (2.20) and expanding, we obtain the generalised stiffness matrix which takes into account the moment rotation terms as follows.

$$\begin{bmatrix} H_1 \\ V_1 \\ M_1 \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & -k_{11}\frac{h}{2} & \cdot & \cdot & \cdot \\ k_{21} & k_{22} & -k_{21}\frac{h}{2} & \cdot & \cdot & \cdot \\ -k_{11}\frac{h}{2} & -k_{12}\frac{h}{2} & k_{11}\frac{h^2}{4} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ \theta_x \\ \cdot \\ \cdot \end{bmatrix} \quad (2.22)$$

This equation for generalised foundation model can be modified to obtain either a one parameter model (Winkler foundation) or a two parameter model with interactions between vertical displacements by supressing the unnecessary terms.

2.6 Overall Stiffness Matrix:

The overall stiffness matrix for the laminate resting on elastic foundation is obtained by adding the foundation stiffness matrix to that of the laminate.

CHAPTER 3

FINITE ELEMENT FORMULATION

3.1 Introduction:

In general plate bending analysis in the case of isotropic plates is done by considering the transverse displacement and the two shear rotations as field variables. In the case of fiber composites, the transverse shear effects are more pronounced in comparison with the isotropic case as the transverse shear modulus is very small compared to the inplane Young's modulus. The inplane displacements have a significant effect on the natural frequencies of composite laminates. In this chapter a finite element model for the dynamic analysis of composite laminates is developed. The effect of elastic foundation is incorporated in the plate stiffness matrix.

3.2 The Element:

An isoparametric quadrilateral plate bending element with u_o, v_o, w, Ψ_x, Ψ_y as the field variables over the element is chosen for the analysis (Figure 3.1).

3.2.1 Shape functions:

Shape functions associated with each node define,

(i) geometry of the element in terms of nodal coordinates

(ii) displacement variation over the element in terms of nodal displacements.

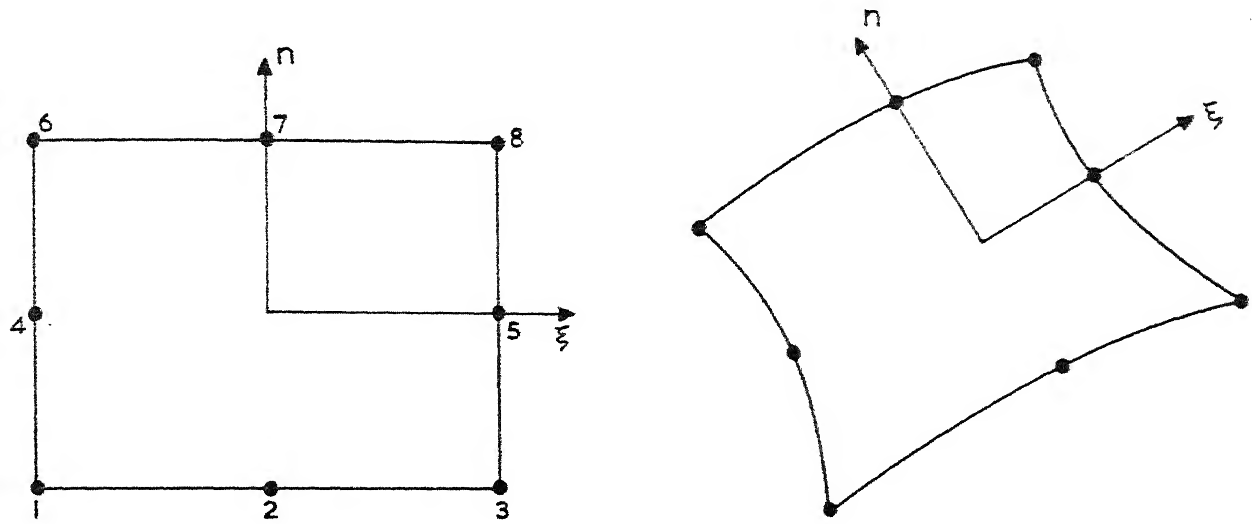

FIG. 3.1 PARABOLIC ISOPARAMETRIC PLATE BENDING ELEMENT.

At any point within the element

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{r=1}^{8} [N_r] \{\phi_r\} \tag{3.1}$$

$$\{u\} = \sum_{r=1}^{8} [N_r] \{\delta_r\} \tag{3.2}$$

The shape functions associated with each node for the element in terms of non dimensionalised coordinates ξ, η are as follows.

$$\begin{aligned}
N_1 &= \tfrac{1}{4}(1 - \xi)(1 - \eta)(1 - \xi - \eta) \\
N_2 &= \tfrac{1}{2}(1 - \xi^2)(1 - \eta) \\
N_3 &= \tfrac{1}{4}(1 + \xi)(1 - \eta)(- 1 + \xi - \eta) \\
N_4 &= \tfrac{1}{2}(1 - \xi)(1 - \eta^2) \\
N_5 &= \tfrac{1}{2}(1 + \xi)(1 - \eta^2) \\
N_6 &= \tfrac{1}{4}(1 - \xi)(1 + \eta)(- 1 - \xi + \eta) \\
N_7 &= \tfrac{1}{2}(1 - \xi^2)(1 + \eta) \\
N_8 &= \tfrac{1}{4}(1 + \xi)(1 + \eta)(- 1 + \xi + \eta)
\end{aligned} \tag{3.3}$$

Same shape functions are employed for all the five nodal displacements.

3.3 Elemental Matrices:

3.3.1 Strain displacement relationships:

The generalised strain vector in terms of displacements from the constitutive equation (2.17) is

$$\{\varepsilon\}^t = \{\frac{\partial u_o}{\partial x}, \frac{\partial v_o}{\partial y}, \frac{\partial w}{\partial y} + \Psi_y, \frac{\partial w}{\partial x} + \Psi_x, \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}, \frac{\partial \Psi_x}{\partial x}, \frac{\partial \Psi_y}{\partial y}, \frac{\partial \Psi_x}{\partial y} + \frac{\partial \Psi_y}{\partial x}\} \tag{3.4}$$

In terms of nodal displacements,

$$\{\varepsilon\} = \sum_{r=1}^{8} [B_r]\{\delta_r\} \tag{3.5}$$

where

$$[B] = [L][N] \tag{3.6}$$

L is the linear operator matrix obtained from the nodal strain vector (3.4) as

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 1 \\ 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{3.7}$$

3.3.2 Stiffness matrix:

Stiffness matrix is given by [24]

$$[k^e] = \int_{-1}^{+1}\int_{-1}^{+1} [B]^t [D] [B] \, d\xi \, d\eta \, \det J \qquad (3.8)$$

with det J accounting for the transformation of the coordinates from local to global and J is defined by

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad (3.9)$$

with

$$\frac{\partial x}{\partial \xi} = \sum_{r=1}^{8} \frac{\partial N_r}{\partial \xi} x_r , \qquad \frac{\partial x}{\partial \eta} = \sum_{r=1}^{8} \frac{\partial N_r}{\partial \eta} x_r$$

$$\frac{\partial y}{\partial \xi} = \sum_{r=1}^{8} \frac{\partial N_r}{\partial \xi} y_r , \qquad \frac{\partial y}{\partial \eta} = \sum_{r=1}^{8} \frac{\partial N_r}{\partial \eta} y_r \qquad (3.10)$$

[D] is the material property matrix occurring in the stress strain relationships.

The numerical integration in equation (3.8) is carried out by 2 x 2 Gauss quadrature [11].

3.3.3 Mass matrix:

Consistent mass matrix is derived using the same interpolation functions as given in equation (3.3).

Nodal accelerations at any node r are given by

$$\{\ddot{\delta}_r\}^t = \{\ddot{u}_o, \ddot{v}_o, \ddot{w}, \ddot{\Psi}_x, \ddot{\Psi}_y\} \qquad (3.11)$$

$\ddot{u}_o$, $\ddot{v}_o$ and $\ddot{w}$ produce inplane and lateral inertia forces respectively and $\ddot{\Psi}_x$, $\ddot{\Psi}_y$ produce rotary inertia couples. The inertia force vector can be written as

$$\{a\}^t = \{\rho\,\ddot{u}_o,\ \rho\,\ddot{v}_o,\ \rho\,\ddot{w},\ \frac{\rho h^2}{12}\,\ddot{\Psi}_x,\ \frac{\rho h^2}{12}\,\ddot{\Psi}_y\} \qquad (3.12)$$

At any point in the element, inertia force can be expressed as

$$\{a\} = [P]\,[N]\,\{\ddot{\delta}^e\} \qquad (3.13)$$

where

$$\{\ddot{\delta}^e\}^t = \{\ddot{\delta}_1,\ \ddot{\delta}_2\ \cdot\ \cdot\ \cdot\ \cdot\ \ddot{\delta}_8\}$$

and

$$[P] = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ & \rho & 0 & 0 & 0 \\ & & \rho & 0 & 0 \\ & \text{sym} & & \frac{\rho h^2}{12} & 0 \\ & & & & \frac{\rho h^2}{12} \end{bmatrix} \qquad (3.14)$$

Nodal forces due to inertia can be expressed as

$$F_i^e = \int [N]^t\,\{a\}\,dA \qquad (3.15)$$

Using (3.13) in (3.15)

$$= \int [N]^t\,[P]\,[N]\,dA\,\{\ddot{\delta}^e\} \qquad (3.16)$$

$$= [M^e]\,\{\ddot{\delta}^e\} \qquad (3.17)$$

where

$[M^e]$ is the elemental consistent mass matrix given by

$$[M^e] = \int_{-1}^{+1} \int_{-1}^{+1} [N]^t \, [P] \, [N] \, d\xi \, d\eta \det J \qquad (3.18)$$

The consistent mass matrix has been evaluated using 3 x 3 Gauss quadrature rule [11].

3.3.4 Resultant stiffness matrix:

The resultant elemental stiffness matrix is obtained by adding the foundation stiffness of equation (2.20) to the elemental stiffness obtained in equation (3.8).

3.4 Assembly of Elemental Matrices:

Assembly of elemental matrices into global matrices is done by imposing the conditions that the displacement vectors are common and sum of the generalised force vectors are zero at common nodes.

3.5 Solution for Free Response:

With no forcing and damping terms existing, the dynamic problem is defined as

$$[M] \{\ddot{\Delta}\} + [K] \{\Delta\} = 0 \qquad (3.19)$$

By substituting $\{\Delta\} = \{\bar{\Delta}\} e^{i\omega t}$ in (3.19) we obtain the typical eigen value problem

$$([K] - \omega^2 [M]) \{\bar{\Delta}\} = 0 \qquad (3.20)$$

For the non zero solutions of the above eigen problem, the determinant of the equation (3.20) must be zero.

$$|[K] - \omega^2 [M]| = 0 \tag{3.21}$$

This equation yields the n values of ω^2 when the size of matrices [K] and [M] is n x n. Usually, in structural problems [K] and [M] are positive definite thereby resulting in positive eigen values.

CHAPTER 4

RESULTS

4.1 Introduction:

A computer program based on the finite element formulation presented in Chapter 3 is developed to investigate the effects of elastic foundation on the fundamental natural frequencies of isotropic plates and orthotropic laminates. Two types of orthotropic plates are used. Biaxial symmetry of the plate is made use of and quarter plate model is used to compute the fundamental frequencies and a half plate model is used to compute frequencies at higher modes. The mesh is of the order 2 x 2 for the quarter plate and that for the half plate is 2 x 4. Simply supported boundary conditions are shown in Figure 4.1.

4.1.1 Material properties:

Following material properties are used.

Isotropic case:

$$E = 30 \times 10^6 \text{ psi}$$

$$\nu = 0.3$$

$$\rho = 0.283 \text{ lb/in}^3$$

Orthotropic case (graphite epoxy):

$$E_L = 27 \times 10^6 \text{ psi}$$

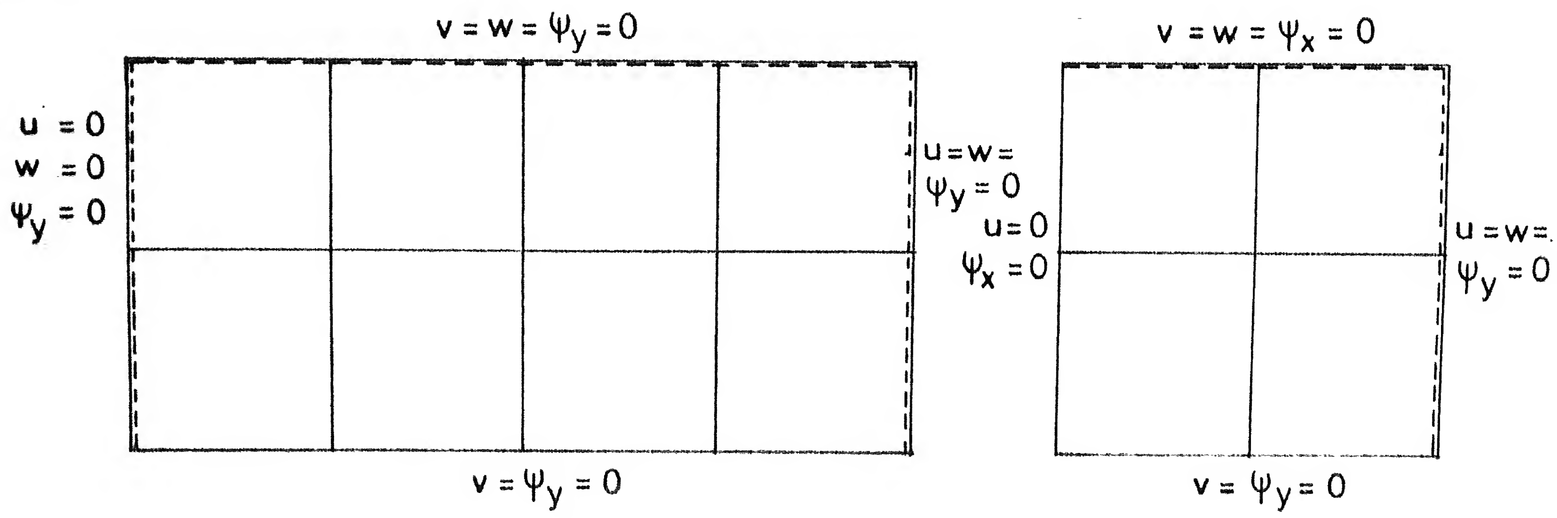

FIG. 4.1 SIMPLY SUPPORTED BOUNDARY CONDITIONS FOR HALF AND QUARTER PLATES.

$$E_T = 0.675 \times 10^6 \text{ psi}$$

$$G_{LT} = 0.405 \times 10^6 \text{ psi}$$

$$\nu_{LT} = 0.25$$

$$\rho = 0.056 \text{ lb/in}^3$$

Four layers of equal thickness made of same orthotropic material is assumed to constitute the laminate.

Calculation of D matrix is based on the equation (2.17) for orthotropic case and the standard stress strain relationships are used for the isotropic case.

4.1.2 Accuracy of the model:

Numerical accuracy of the program developed is established by comparing the results obtained in earlier works 6, 11, 17 . Comparisons in respect of following are made.

i) fundamental frequencies of a simply supported, square isotropic plate (Table 1)

ii) fundamental frequencies of square simply supported orthotropic laminate (Table 2)

iii) static deflection of central point of square isotropic plate on Winkler foundation (Table 3).

In the third case, the generalised foundation model is reduced to that of a Winkler type and the results are compared.

All tabulated frequencies are non dimensionalised as shown.

4.2 Isotropic Plate:

One parameter, two parameter and generalised foundation models are used to investigate the effects of elastic foundation

Table 1. Isotropic non dimensional frequencies

$$\Omega = \omega\left(\frac{\rho h^2}{G}\right)^{1/2} \quad \frac{a}{b} = 1\,, \quad \frac{a}{h} = 10, \text{ simply supported}$$

Mode	Mindlin [6]	Rock & Hinton [12]	Present model
1, 1	0.0930	0.0931	0.09313
2, 1	0.2217	0.2237	0.2239
3, 1	0.4144	0.4312	0.4317
3, 2	0.5197	0.5379	0.5500

Table 2. Orthotropic non dimensional frequencies $\Omega = \omega a^2 (\frac{\rho}{E_T h^2})^{1/2}$, simply supported plate

θ	30°		45°	
a/h \ a/b	1.0	1.2	1.0	1.2
10	17.616 *(17.689)	19.810 (19.880)	18.360 (18.609)	21.496 (21.567)
20	21.344 (21.281)	24.427 (24.362)	22.385 (22.584)	26.877 (26.837)

* Ref. [13].

Table 3. Central point displacements of simply supported square isotropic plate on Winkler foundation

E/E_f \ h/a	0.03	0.05	0.10
10	0.56×10^{-6} *(0.79×10^{-6})	0.41×10^{-6} (0.49×10^{-6})	0.28×10^{-6} (0.25×10^{-6})
100	0.37×10^{-5} *(0.325×10^{-5})	0.218×10^{-5} (0.22×10^{-5})	0.1025×10^{-5} (0.120×10^{-5})

* Ref. [18].

on the fundamental natural frequencies. The results obtained by using the two parameter and generalised foundation models are found to almost coincide.

The fundamental natural frequencies are found to increase with the increase in foundation stiffness. At values of $\frac{E}{E_f}$ smaller than 100 (very stiff foundation) the fundamental frequency increases sharply as shown in Figure 4.1

The mode shapes are investigated and it is observed that all the modes are predominantly bending modes. The results obtained by using the two parameter foundation model are lower than those obtained by the one parameter model.

4.3 Symmetric Laminate:

A symmetric orthotropic laminate of the type $[\theta]_4$ is used to investigate the effects of varying ply angle and the ratio E_L/E_f on the fundamental natural frequencies of a simply supported case. For relatively soft foundations the fundamental frequencies are found to increase with increasing ply angle from 0° to 45° and are symmetric with respect to the 45° ply. The frequencies increase with decreasing E_L/E_f ratio. The bending modes dominate in softer foundations as well as in the stiffer foundations ($\frac{E_L}{E_f} < 100$). However at E_L/E_f ratio greater than 50, the fundamental frequency at 45° ply angle is found to have a smaller natural frequency than those at other ply orientations. The smallest frequencies do not vary with further increase in

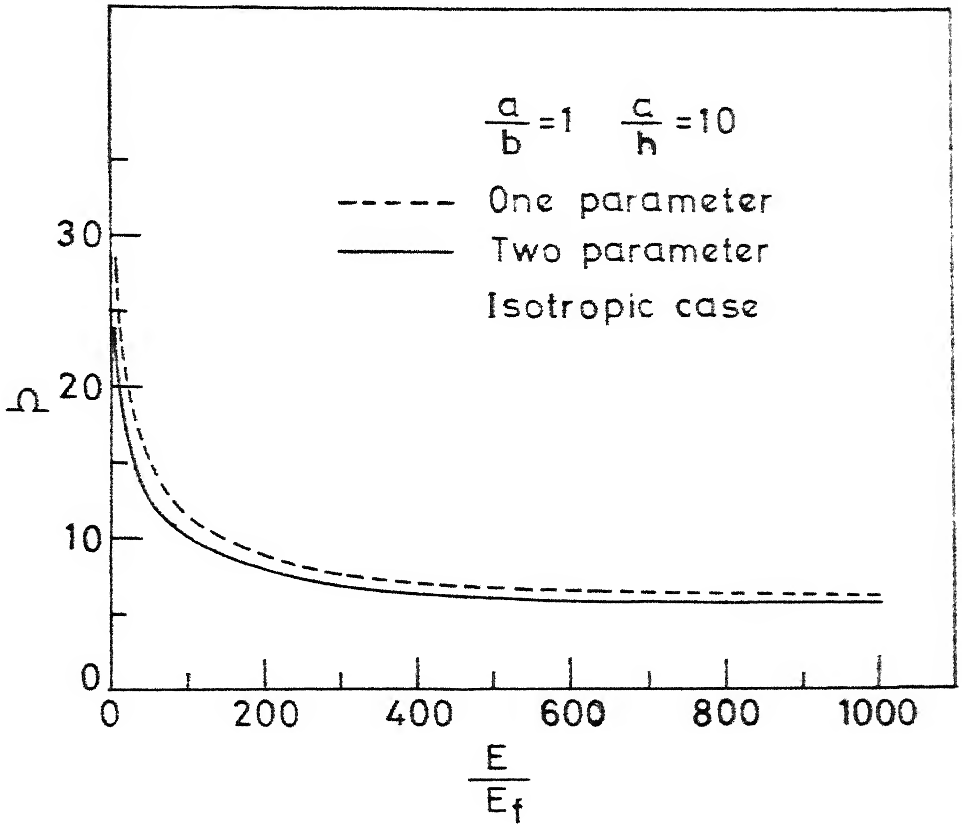

FIG. 4. EFFECT OF FOUNDATION MODULUS ON FREQUENCY.

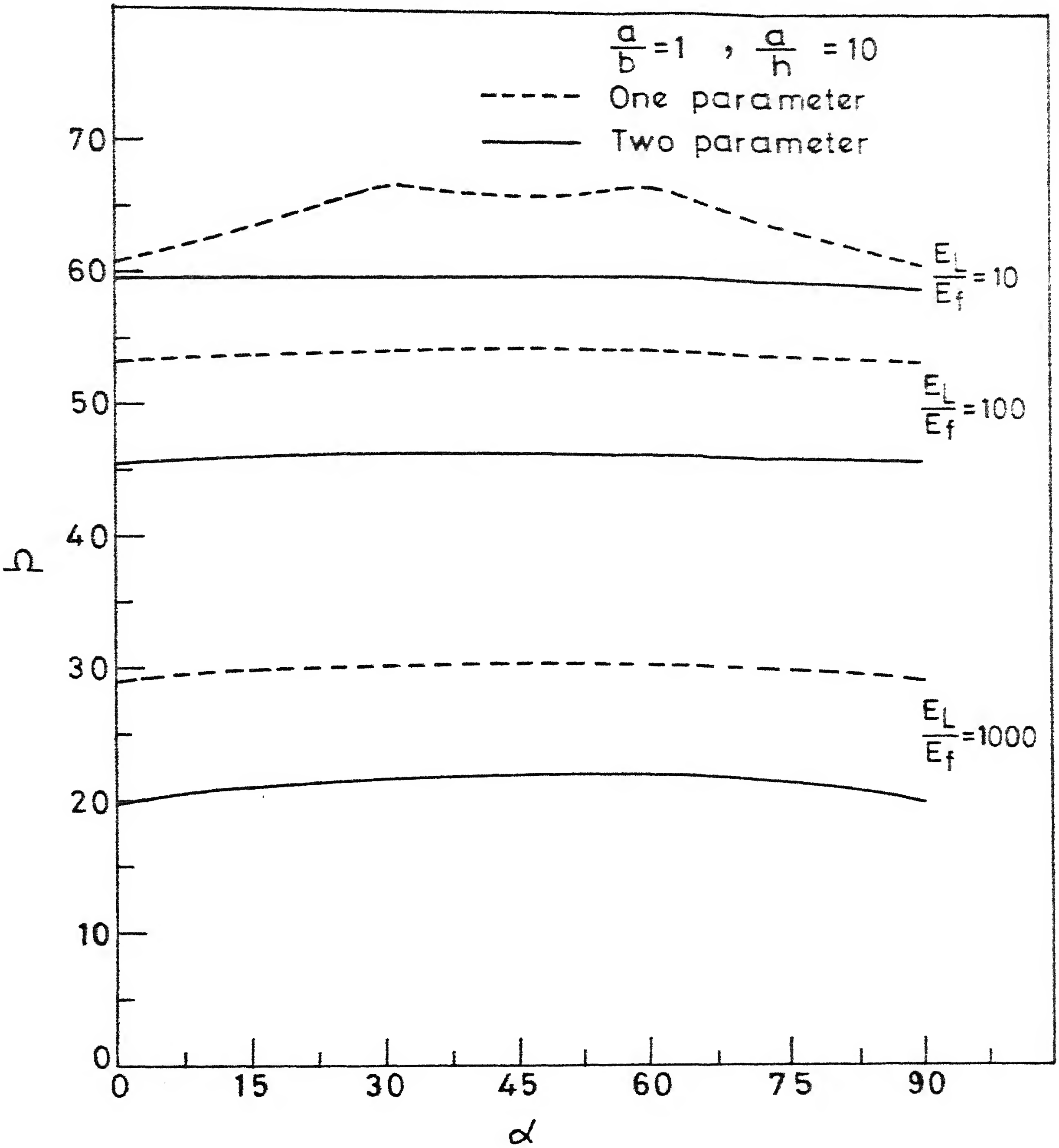

FIG.4.2 EFFECT OF PLY ANGLE ON FREQUENCY SYMM-LAMINATE.

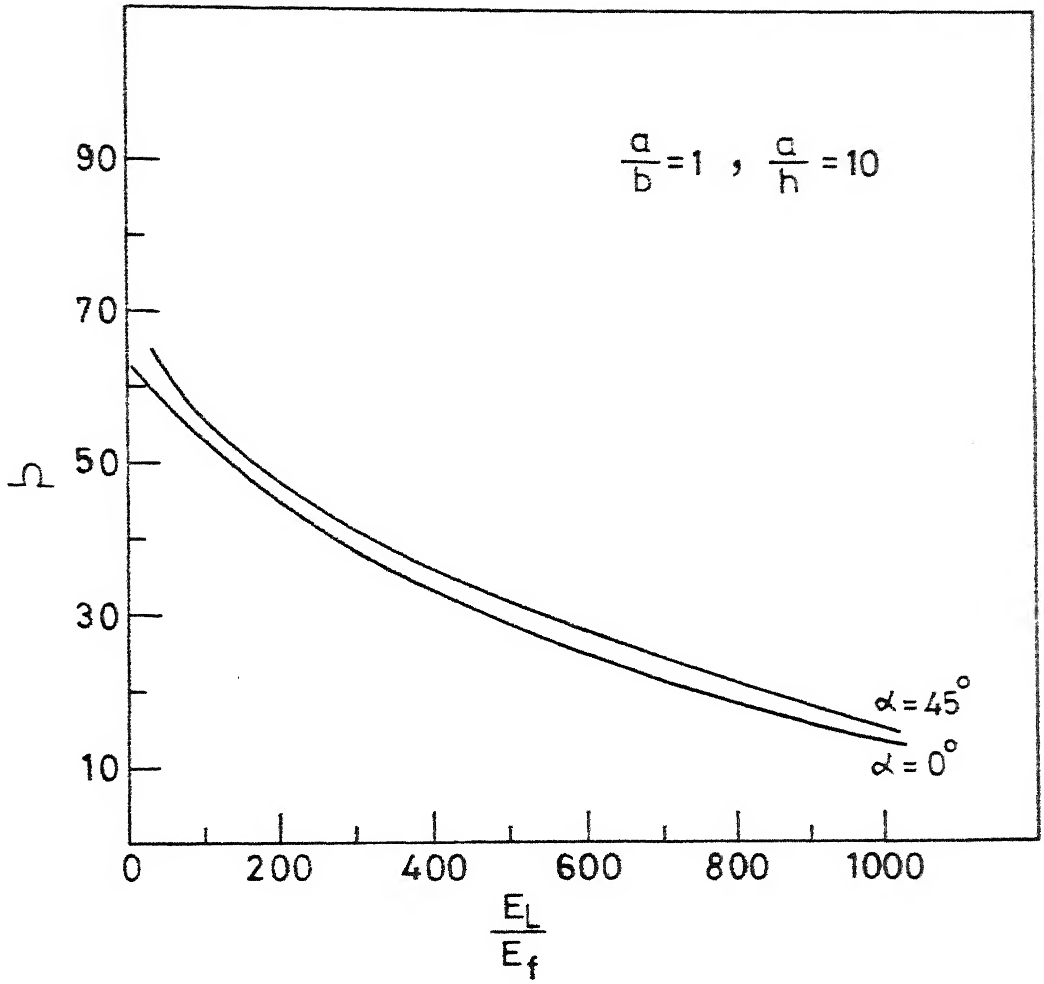

FIG.4.3 EFFECT OF FOUNDATION MODULUS ON FREQUENCY SYMMETRIC LAMINATE.

the foundation stiffness. The results are plotted in Figures 4.2 and 4.3.

4.4 Antisymmetric Laminate:

An antisymmetric angle ply laminate of configuration $[+\theta / -\theta / +\theta / -\theta]_T$ is chosen. As in the previous cases, investigations were carried out for a square simply supported laminate.

At lower foundation stiffness (higher values of E_L/E_f) the natural frequencies vary as in the case of the symmetric laminate. The predominant mode is the bending mode. With stiffer foundations the stretching mode is excited due to higher bending stiffnesses. However, a sharp variation in fundamental frequencies is observed around E/E_f ratio of 10. The highest frequency belonging to the 30° ply orientation and the smallest belonging to the 45° ply orientation. These frequencies **correspond** to the stretching modes. The smallest frequency in the stretching mode at 45° orientation can be attributed to the smallest values of E_L and E_T (in plane moduli) at 45° ply orientation. However, this behaviour is restricted to this type of laminates only and separate investigations are necessary for different types of laminates as the laminate stiffness itself is affected by the change of any of the laminate parameters. The behaviour is plotted in Figure 4.4.

The parametric effects are tabulated by varying the aspect ratio and side to thickness ratio at different ply angles and the results are tabulated in Table 4. The variation

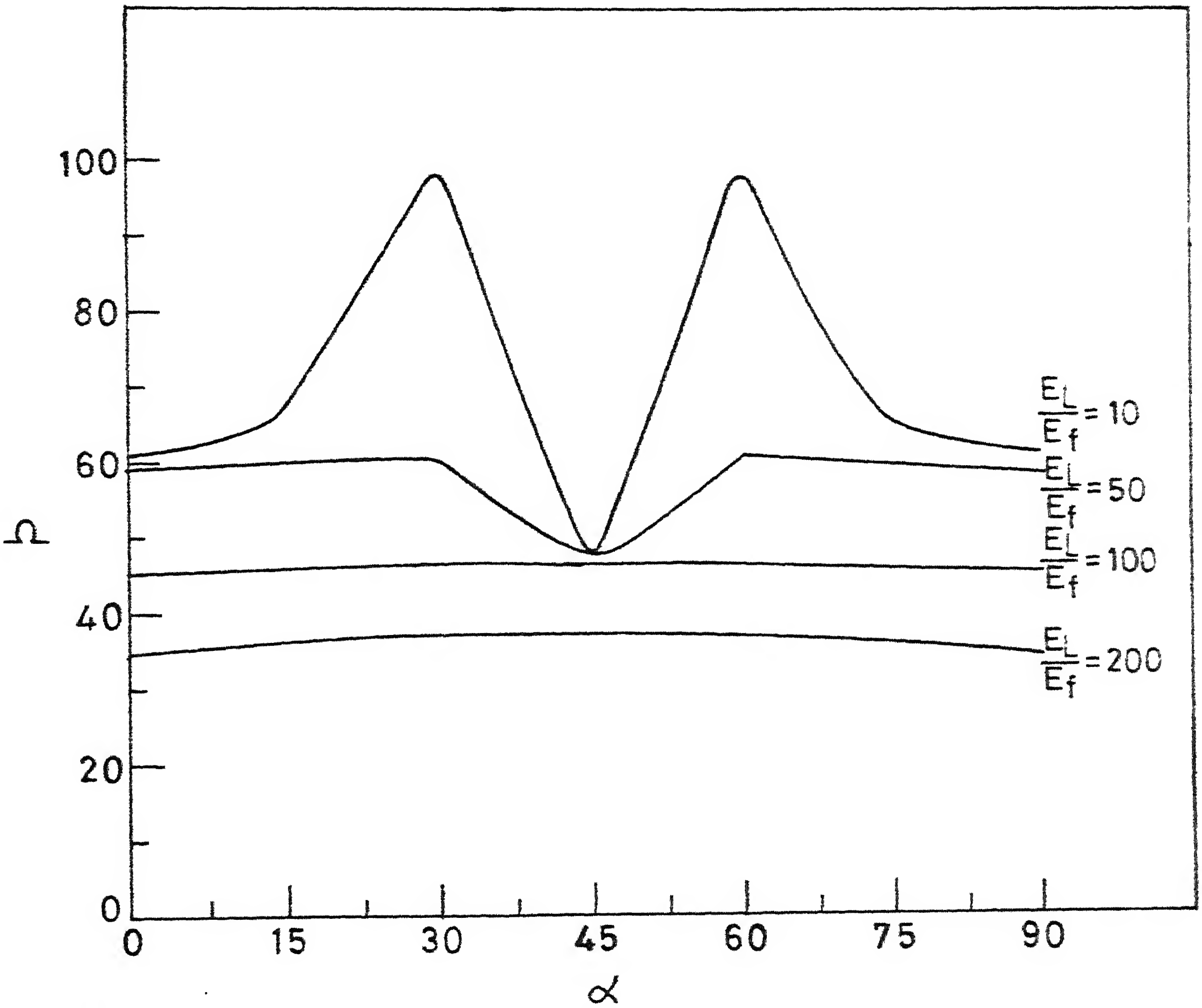

FIG. 4.4 EFFECT OF PLY ANGLE AND FOUNDATION STIFFNESS

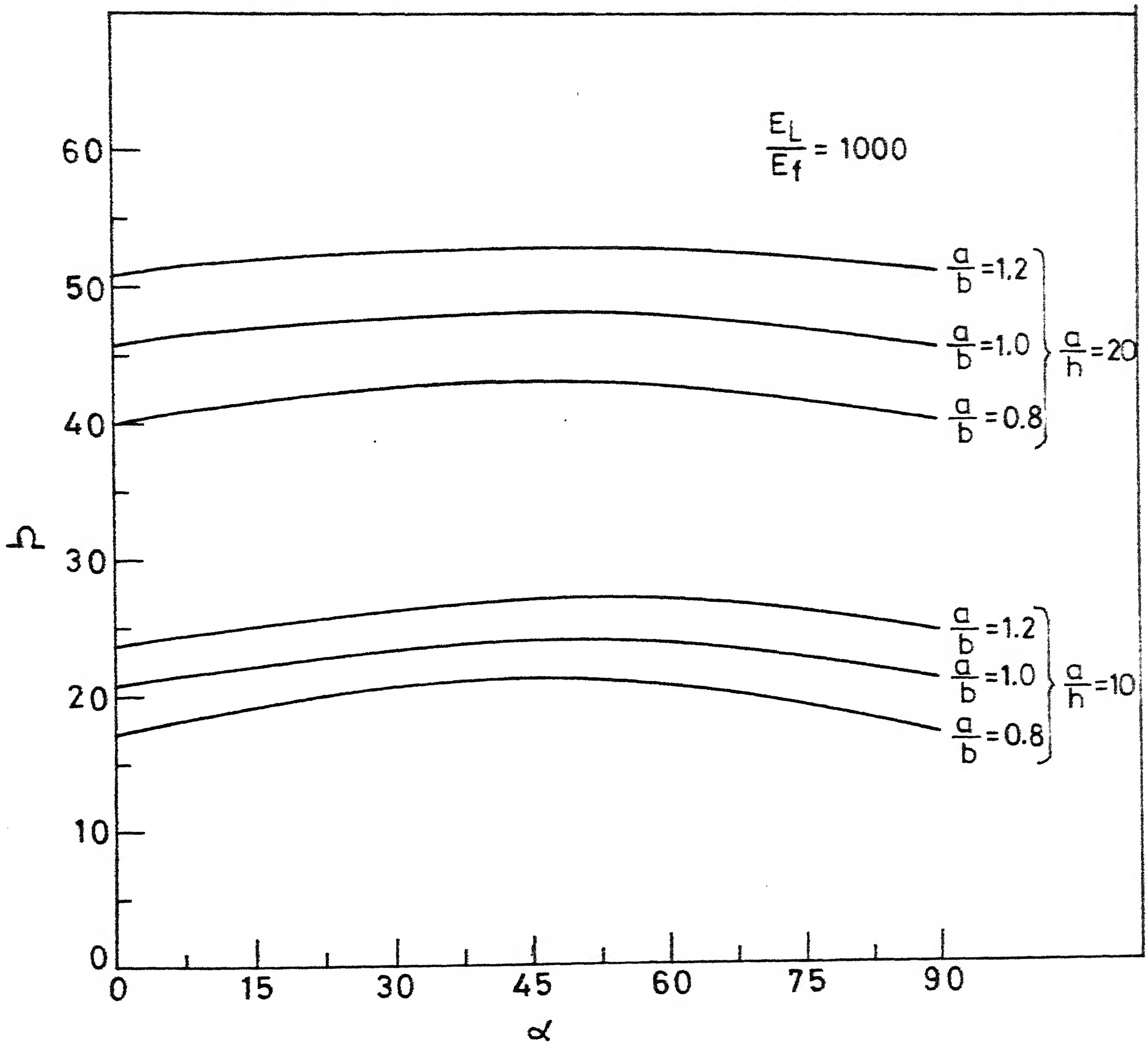

FIG.4.5 PARAMETRIC EFFECTS-ANTISYMMETRIC LAMINATE.

Table 4. Parametric effects: simply supported antisymmetric angle ply laminate $\Omega = \omega a^2\left(\frac{\rho}{E_T h^2}\right)^{1/2}$ (2 parameter foundation)

θ	E_L/E_f	10			100			1000		
	a/h \ a/b	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2
30°	10	82.66	97.53	89.41	42.65	42.20	51.26	20.77	23.25	25.70
	20	168.65	198.73	183.37	105.57	117.34	127.68	42.78	47.50	51.86
	30	155.78	255.57	301.11	180.51	201.58	220.27	72.06	79.88	86.76
45°	10	59.15	47.54	60.38	42.69	45.89	51.91	20.85	23.79	27.01
	20	126.15	107.72	134.76	106.26	105.16	128.40	42.831	47.99	53.07
	30	196.43	173.52	215.38	182.68	170.31	211.85	72.42	80.41	87.70

of fundamental frequencies at different ply angles and parametric ratios are plotted in Figure 4.5.

4.5 Conclusions:

Since the laminates can be made to suit the requirements of desired anisotropy, separate analysis wherever is required is to be carried out to ascertain the dynamic behaviour of the particular type of laminate. An extension of the present work can be undertaken for non linear vibration analysis including damping and for other boundary and external loading conditions. Experimental work to ascertain the behaviour of laminates on elastic foundations may be undertaken. It is hoped that the present work can form a basis for undertaking the dynamic analysis of plates that can incorporate the interaction of the inertia of foundation system with that of the structure.

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